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DETERMINATION OF THE TEMPERATURE-DEPENDENT VARIATION OF THE THERMAL CONDUCTIVITY OF A COMPOSITE MATERIAL FROM THE DATA OF A NONSTATIONARY EXPERIMENT

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We consider the construction of an iteration algorithm for reconstructing the temperature-dependent variation of the thermal conductivity in the generalized energy equation from the data of temperature measurements at one or more points in the interior of the body.

In investigating the thermophysical characteristics of composite materials, it often becomes necessary to use new methods for the analysis and processing of experimental data. These methods must provide a possibility of processing the results of a nonstationary thermal experiment and obtaining the maximum amount of reliable information concerning the material under study when the accuracy characteristics of the measurement systems are limited [1].

The intensive development of the theory and the expansion of the fields of practical application of methods for the solution of inverse problems in heat exchange have led to their widespread use in thermophysical investigations.

A particularly timely use of the inverse-problem apparatus is its application to the investigation of the thermophysical characteristics of high-temperature composite materials under nonstationary conditions. Such an approach enables us to eliminate the problem of simulating the structure of the material and the character of the internal processes under nonstationary thermal influences. Furthermore, in this case there is a possibility of considering the problem of thermophysical investigations as a complex problem in the simultaneous determination of many interrelated characteristics.

Inverse problems usually belong to the class of ill-posed problems of mathematical physics [2]. In solving boundary-value and coefficient-type inverse problems in heat conduction, iterative methods have been found to be very effective [3-5].

The basic purpose of the present study is to investigate the possibilities of constructing iteration algorithms of the gradient type for reconstructing the thermophysical characteristics of a composite material from the solution of a coefficient-type inverse problem for the nonlinear generalized heat-conduction equation. We analyze a mathematical model which takes account of the processes of thermal decomposition and filtration [1].

We shall consider the following problem. For a given mathematical model of the process of heat and mass exchange during the intense heating of a composite heat-shielding material and known boundary conditions, it is required to reconstruct the temperature-dependent variation of the thermal conductivity $\lambda(T)$ and the temperature field $T(x, \tau)$ from the data of temperature measurements at one or more interior points of the body under investigation. The mathematical model of the process being investigated has the following form:

$$c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) - m_g\frac{\partial h_g(T)}{\partial T}\frac{\partial T}{\partial x} - \frac{\partial m_g}{\partial x}h_g(T),$$

$$0 < x < b,\ 0 < \tau \leqslant \tau_m, \tag{1}$$

$$T(0,\ \tau) = f_1(\tau), \tag{2}$$

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$$T(b, \tau) = f_N(\tau), \tag{3}$$

$$T(x, 0) = \varphi(x), \ 0 \leqslant x \leqslant b, \tag{4}$$

$$\frac{dz}{d\tau} = -Az^n \exp\left(-\frac{E}{RT}\right), \ T \geqslant T_0, \tag{5}$$

$$\frac{dz}{d\tau} = 0, \ T < T_0, \tag{6}$$

$$\frac{\partial m_g}{\partial x} = -(1 - k_T)\rho_0 \frac{dz}{d\tau}, \tag{7}$$

$$m_g = \int_0^x \frac{\partial m_g}{\partial x} dx, \ 0 \leqslant x \leqslant b, \tag{8}$$

$$T(x_i, \tau) = f_i(\tau), \ 0 < x_i < b, \ i = 2, 3, \ldots, N-1, \tag{9}$$

where $c(T)$, $h_g(T)$, $\varphi(x)$, $f_i(\tau)$, $i = 1, 2, 3, \ldots, N$, are known functions. The parameters in the expressions (5)-(7) describing the process of thermal degradation in the composite material are assumed to be given.

Using the parametrization of the desired function $\lambda(T)$, we can reduce the initial nonlinear variational problem to an extremal problem which consists in finding a vector of parameters that minimizes the selected quality criterion. A very convenient type of parametrization is the approximation of the functions by B-splines [6]. In this case the thermal conductivity can be represented as a function of the temperature in the form

$$\lambda(T) = \sum_{j=-1}^{m+1} \lambda_j B_j(T), \tag{10}$$

where λ_j are the desired coefficients of the spline; $B_j(T)$ are the B-splines.

The inverse problem formulated in this manner will be treated as an optimal-control problem, where the control influence is the vector of parameters $\bar{\lambda} = \{\lambda_{-1}, \lambda_0, \ldots, \lambda_{m+1}\}$. As the target functional we select the rms deviation

$$I(x, \tau, \lambda(T)) = \sum_{i=2}^{N-1} \int_0^{\tau_m} [T_i(x, \tau, \lambda(T)) - f_i(\tau)]^2 d\tau. \tag{11}$$

The solution of the extremum problem (1)-(11) will be constructed by using gradient methods of minimization. To do this, we obtain formulas for the components of the gradient of the functional (11) in terms of the desired parameters.

If measurements of the temperature are made in the interior of the body under investigation, it is convenient to represent the system of equations (1)-(8) as a problem in the heating of an unbounded multilayer plate in which the layers have identical thermophysical properties. This approach enables us, in the numerical integration of the system (1)-(8), to introduce a nonuniform network in the space coordinate. Assuming that there is ideal contact between the individual layers, we find

$$c(T)\frac{\partial T_i}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T_i}{\partial x}\right) - m_g \frac{\partial h_g(T)}{\partial T_i}\frac{\partial T_i}{\partial x} - \frac{\partial m_g}{\partial x} h_g(T),$$

$$0 < \tau \leqslant \tau_m, \ i = 1, 2, \ldots, N-1, \ x_1 = 0, \ x_N = b, \tag{12}$$

$$T_1(0, \tau) = f_1(\tau), \tag{13}$$

$$\left.\begin{aligned} &T_{i-1}(x_i, \tau) = T_i(x_i, \tau), &(14)\\ &\frac{\partial T_{i-1}(x_i, \tau)}{\partial x} = \frac{\partial T_i(x_i, \tau)}{\partial x}, &(15)\end{aligned}\right\} \ i = 2, 3, \ldots, N-1,$$

$$T_{N-1}(b, \tau) = f_N(\tau), \tag{16}$$

$$T_i(x, 0) = \varphi_i(x), \ i = 1, 2, \ldots, N-1, \tag{17}$$

$$\frac{dz}{d\tau} = -Az^n \exp\left(-\frac{E}{RT_i}\right), \ T_i \geqslant T_0, \tag{18}$$

$$\frac{dz}{d\tau} = 0, \; T_i < T_0, \tag{19}$$

$$\frac{\partial m_g}{\partial x} = -(1 - k_T)\rho_0 \frac{dz}{d\tau}, \tag{20}$$

$$m_g = \int\limits_0^x \frac{\partial m_g}{\partial x} dx, \; 0 \leqslant x \leqslant b, \tag{21}$$

$$T_i(x_i, \tau) = f_i(\tau), \; x_1 < x_i < x_N, \; i = 2, 3, \ldots, N-1. \tag{22}$$

The inverse problem is that of determining the vector $\overline{\lambda} = \{\lambda_{-1}, \lambda_0, \ldots, \lambda_{m+1}\}$ which minimizes the functional (11), taking account of the conditions (12)-(22).

If the components of the desired vector $\overline{\lambda}$ are given small increments $\Delta\lambda_j$, $j = -1, 0, \ldots, m + 1$, then the temperature $T_i(x, \tau)$ will show an increment $\vartheta_i(x, \tau)$. Making use of the system (12)-(21), we can show that in the linear approximation the function $\vartheta_i(x, \tau)$ satisfies the following boundary-value problem:

$$c\frac{\partial \vartheta_i}{\partial \tau} = \lambda \frac{\partial^2 \vartheta_i}{\partial x^2} + \left(2\frac{\partial \lambda}{\partial T_i}\frac{\partial T_i}{\partial x} - k\right)\frac{\partial \vartheta_i}{\partial x} + \left[\frac{\partial^2 \lambda}{\partial T_i^2}\left(\frac{\partial T_i}{\partial x}\right)^2 + \frac{\partial \lambda}{\partial T_i}\frac{\partial^2 T_i}{\partial x^2} - \frac{\partial k}{\partial x} - \frac{\partial Q}{\partial T_i} + \right.$$

$$\left. + \frac{\partial c}{\partial T_i}\frac{\partial T_i}{\partial \tau}\right]\vartheta_i + \frac{\partial (\Delta\lambda)}{\partial T_i}\left(\frac{\partial T_i}{\partial x}\right)^2 + \Delta\lambda\frac{\partial^2 T_i}{\partial x^2},$$

$$0 < \tau \leqslant \tau_m, \; x_i < x < x_{i+1}, \; i = 1, 2, \ldots, N-1,$$

$$x_1 = 0, \; x_N = b, \tag{23}$$

$$\vartheta_1(0, \tau) = 0, \tag{24}$$

$$\left.\begin{aligned} &\vartheta_{i-1}(x_i, \tau) = \vartheta_i(x_i, \tau), && (25)\\ &\frac{\partial \vartheta_{i-1}(x_i, \tau)}{\partial x} = \frac{\partial \vartheta_i(x_i, \tau)}{\partial x}, && (26)\end{aligned}\right\} \; i = 2, 3, \ldots, N-1,$$

$$\vartheta_{N-1}(b, \tau) = 0, \tag{27}$$

$$\vartheta_i(x, 0) = 0, \; i = 1, 2, \ldots, N-1, \tag{28}$$

where

$$k = m_g \frac{\partial h_g(T)}{\partial T_i}; \; Q = h_g(T)\frac{\partial m_g}{\partial x}.$$

To calculate the coefficients k and Q in Eq. (23) we use relations (18)-(21).

We write the linear part of the increment of the target functional:

$$\Delta I = 2\sum_{i=2}^{N-1}\int\limits_0^{\tau_m}[T_i(x_i, \tau) - f_i(\tau)]\,\vartheta_i(x_i, \tau)\,d\tau. \tag{29}$$

It should be noted that in the linear approximation the boundary-value problem for the function $\vartheta_i(x, \tau)$ in the form (23)-(28) is also valid for the case in which T_0 depends on the rate of heating $\partial T(x, \tau)/\partial \tau$.

Introducing the boundary-value problem conjugate to system (12)-(22) enables us to obtain analytic expressions for the components of the gradient of the target functional. In this case the boundary-value problem for the conjugate variable $\psi_i(x, \tau)$ has the form

$$-c\frac{\partial \psi_i}{\partial \tau} = \lambda\frac{\partial^2 \psi_i}{\partial x^2} + k\frac{\partial \psi_i}{\partial x} - \frac{\partial Q}{\partial T_i}\psi_i,$$

$$0 < \tau \leqslant \tau_m, \; x_i < x < x_{i+1}, \; i = 1, 2, \ldots, N-1, \tag{30}$$

$$\psi_1(0, \tau) = 0, \tag{31}$$

$$\left.\begin{aligned} &\psi_{i-1}(x_i,\ \tau)=\psi_i(x_i,\ \tau), &(32)\\ &\frac{\partial\psi_{i-1}(x_i,\ \tau)}{\partial x}-\frac{\partial\psi_i(x_i,\ \tau)}{\partial x}=\\ &=\frac{2}{\lambda}[T_i(x_i,\ \tau)-f_i(\tau)], &(33)\end{aligned}\right\}\ i=2,\ 3,\ \ldots,\ N-1,$$

$$\psi_{N-1}(b,\ \tau)=0, \tag{34}$$

$$\psi_i(x,\ \tau_m)=0,\ i=1,\ 2,\ \ldots,\ N-1. \tag{35}$$

Making use of the relations (33), (31), and (26) as well as Eqs. (23) and (30) and Eqs. (35), (27), and (32), after some transformations, we obtain an expression for the increment of the functional

$$\Delta I=\sum_{i=1}^{N-1}\int_0^{\tau_m}\int_{x_i}^{x_{i+1}}\psi_i\left[\frac{\partial(\Delta\lambda)}{\partial T_i}\left(\frac{\partial T_i}{\partial x}\right)^2+\Delta\lambda\,\frac{\partial^2T_i}{\partial x^2}\right]dxd\tau. \tag{36}$$

Taking into account the representation of the desired function in the form (10), we have the following expressions for the components of the target functional vector:

$$I'_{\lambda_j}=\frac{\partial I}{\partial\lambda_j}=\sum_{i=1}^{N-1}\int_0^{\tau_m}\int_{x_i}^{x_{i+1}}\psi_i\left[\frac{dB_j(T)}{dT_i}\left(\frac{\partial T_i}{\partial x}\right)^2+B_j(T)\,\frac{\partial^2T_i}{\partial x^2}\right]dxd\tau,$$
$$j=-1,\ 0,\ \ldots,\ m+1. \tag{37}$$

Knowing the value of the gradient of the target functional, we construct the process of successive approximations on the basis of the method of conjugate gradients [7]. In this case the approximations are carried out by the formula

$$\overline{\lambda}^{(S+1)}=\overline{\lambda}^{(S)}+\alpha^{(S)}\overline{G}^{(S)},\ S=1,\ 2,\ 3,\ \ldots, \tag{38}$$

where $\overline{\lambda}=\{\lambda_{-1},\ \lambda_0,\ \ldots,\ \lambda_{m+1}\}$, $\overline{G}^{(S)}=-I'_\lambda+\beta^{(S)}\overline{G}^{(S-1)}$,

$$\overline{G}=\{g_{\lambda_{-1}},\ g_{\lambda_0},\ \ldots,\ g_{\lambda_{m+1}}\};\ \beta^{(1)}=0;$$

$$\beta^{(S)}=\frac{\sum_{i=1}^{N-1}\int_0^{\tau_m}I_\lambda'^{(S)}(I_\lambda'^{(S)}-I_\lambda'^{(S-1)})\,d\tau}{\sum_{i=1}^{N-1}\int_0^{\tau_m}(I_\lambda'^{(S-1)})^2d\tau};$$

$\alpha^{(S)}$ is the depth of descent, S is the iteration number.

It can be shown that the problem (23)-(28) is linear in the parameter $\alpha^{(S)}$, i.e.,

$$\vartheta_i(x,\ \tau,\ \alpha^{(S)},\ \Delta\lambda)=\alpha^{(S)}\vartheta_i(x,\ \tau,\ 1,\ \Delta\lambda).$$

The value of the target functional at the (S + 1)-th iteration will be represented in the form

$$I^{(S+1)}=\sum_{i=2}^{N-1}\int_0^{\tau_m}[T_i(x_i,\ \tau,\ \overline{\lambda}^{(S)})+\alpha^{(S)}\vartheta_i(x_i,\ \tau,\ \overline{G}^{(S)})-f_i(\tau)]^2d\tau,$$

where the magnitude of the step $\alpha^{(S)}$ is taken from the condition that I is minimum at each iteration. Differentiating $I^{(S+1)}$ with respect to $\alpha^{(S)}$ and equating the derivative to zero, we obtain a relation for the linear estimate of the descent step:

$$\alpha^{(S)}=-\frac{\sum_{i=2}^{N-1}\int_0^{\tau_m}[T_i(x_i,\ \tau,\ \overline{\lambda}^{(S)})-f_i(\tau)]\,\vartheta_i(x_i,\ \tau,\ \overline{G}^{(S)})\,d\tau}{\sum_{i=2}^{N-1}\int_0^{\tau_m}[\vartheta_i(x_i,\ \tau,\ \overline{G}^{(S)})]^2\,d\tau}. \tag{39}$$

The iteration process is constructed as follows. We are given the initial value of the desired vector, $\overline{\lambda}^{(1)}$, we solve the direct problem (12)-(21), and we determine the temperature

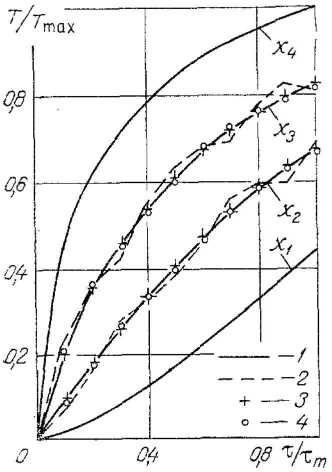

Fig. 1

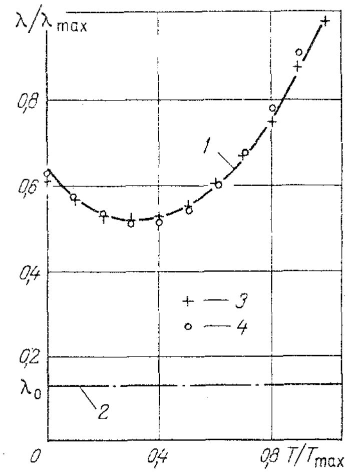

Fig. 2

Fig. 1. Temperature at the points where the thermocouples are set up: 1) exact values; 2) perturbed values; 3, 4) values obtained by solving the inverse problem for the exact and the perturbed data, respectively.

Fig. 2. Reconstruction of the thermal conductivity of a composite material: 1) exact values; 2) initial approximation; 3, 4) solution of the inverse problem for the exact and the perturbed data, respectively.

field. Next we solve the conjugate problem (30)-(35), and by formula (37) we calculate the components of the gradient of the target functional in terms of the desired parameters. After this we solve the problem (23)-(28), and using Eq. (39), we estimate the magnitude of the descent step. The new approximation to the desired vector is determined from the relations (38), and after this the calculation process is repeated. For the case in which we know the exact values of the input temperatures, the iterative process is halted by the condition abs $(T_i - f_i) \leqslant \varepsilon_T$, i.e., i = 2, 3, ..., N − 1, where $\varepsilon_T > 0$ is the error in the calculation of the temperature profile at the points where the thermocouples are set up. In the case when the input temperatures are given with an error, the process is halted by the discrepancy criterion, i.e., when the condition

$$\sum_{i=2}^{N-1} \int_0^{\tau_m} [T_i(x_i, \tau) - f_i(\tau)]^2 d\tau \leqslant \delta^2$$

is satisfied, where $\delta^2 = \sum_{i=2}^{N-1} \int_0^{\tau_m} \sigma_i^2 d\tau$ is the estimate of the generalized error of the initial data; $\sigma_i(\tau)$ is the rms deviation of the input temperatures at the points $x = x_i$ at time τ. With such an approach we can realize a regularized algorithm for the solution of the inverse heat-conduction problem considered here [8].

The above-described algorithm was used as the basis of a FORTRAN program for the BÉSM-6 computer. We used the implicit monotonic approximation scheme of [9] for boundary-value problems on the network

$$\omega = \{x_i = hi,\ i = 0, 1, \ldots, N;\ \tau_j = \Delta\tau j,\ j = 0, 1, \ldots, m\}.$$

As an example illustrating the workability of the proposed algorithm, we considered the inverse problem for the reconstruction of the temperature-dependent variation of the thermal conductivity of a composite heat-shielding material with a silicon-organic resin base.

To describe the nonstationary heating of the material, we used the system of equations (12)-(21). The values of the parameters characterizing the thermal-degradation processes, as well as the functions c(T) and $h_g(T)$, were assumed to be known. The desired function $\lambda(T)$ was approximated by cubic B-splines, and the number of subdivision segments of the maximum temperature interval was taken to be 3.

As the initial data for the solution of the inverse heat-conduction problem, we used the curves of temperature as a function of time obtained from the solution of the direct problem (12)-(21) for boundary conditions of the second kind. Figure 1 shows in dimensionless form the variation of the temperature at the points $x_1 = 0$, $x_2 = 2.1 \cdot 10^{-3}$ m, $x_3 = 2.8 \cdot 10^{-3}$ m, $x_4 = 3.5 \cdot 10^{-3}$ m.

To solve the inverse problem for perturbed data, the deviations of the input temperatures from their exact values were simulated by a random-number generator with a uniform distribution law. The error did not exceed 3% of the maximum value of the temperature.

Figure 2 shows the exact values of the temperature as a function of the thermal conductivity and the values reconstructed from the inverse problem for a composite heat-shielding material. As can be seen, the solution of the inverse problem was obtained with fairly high accuracy, which indicates that it is possible to determine reliable thermophysical characteristics for real materials under nonstationary conditions of thermal loading.

NOTATION

c, volumetric heat capacity; λ, thermal conductivity; m_g, specific mass flow rate; h_g, enthalpy of the gaseous phase; T, temperature; x, coordinate; τ, time; τ_m, the right-hand boundary value of the time interval; $f_i(\tau)$, input temperatures; z, concentration of the decomposable component; A, preexponential multiplier; n, order of the decomposition reaction; E/R, activation energy; k_T, limiting value of the coke number; ρ_0, density of the initial material; T_0, temperature at the beginning of thermal degradation; b, right-hand boundary value of the spatial interval; I, functional; ϑ, increment of temperature; ψ, conjugate variable; i, spatial index; T_{max}, maximum value of the temperature; λ_{max}, maximum value of the thermal conductivity.

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